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Charged Analogue of Whittaker's Interior Solution

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Abstract Charged analogue of Whittaker's interior solution is derived which possess constant gravitational mass density $(\rho + 3p)$. The solution joins smoothly to Nordstörm solution at the pressure free interface. The solution has increasing energy density away from the center which is unphysical however such model can very well represent the situations when density undergoes abrupt changes during the tug of war between the gravitational attraction and electrostatic repulsion.

Keywords Whittaker's solution · Charged analogue

1 Introduction

For last several decades researchers have been deriving solutions for charged fluids to provide source of the Reissner [14] and Nordstörm [13] solutions. Such fluid models are not likely to undergo gravitational collapse to reduce into a point singularity, in presence of charges. The gravitational attraction may be nullified by the electrostatic repulsion and pressure gradient. Several workers, e.g., Bonner [2], Kyle and Martin [12], Cooperstock and Cruz [4], Florides [8], Buchdahl [3], Gupta et al. [9], have obtained charged analogues of Schwarzschild interior solution (with constant energy density). Besides this, a good account of such work is embodied by Ivanov [10, 11].

The present article takes on considerable added interest because of the recent work by Cooperstock and Dupre [5] and Dupre [6] regarding effective mass density which is $(\rho + 3p)$. We present charged analogue of the Whittaker's interior solution possessing constant effective mass density $(\rho + 3p)$. It is obvious that $d\rho/dr > 0$, when $dp/dr < 0$ and such a fluid distribution is not physically realizable and may not lead to a stable model. However, such models are appropriate in the regions in neutron stars where some abrupt changes in energy density take place. This property is also seen in the class of slowly rotating solutions by Wahlquist [17], Bayin [1] and Stewart [15], the first and third being the rotating analogue of the Whittaker's solution.

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2 The Whittaker's Interior Solution

The Schwarzschild interior solution possesses a constant energy density ρ. Whittaker [18] derived one more interior solution with constant gravitational mass density $(\rho + 3p)$. It is the expression, rather than ρ, which governs the gravitational attraction of matter, p being the pressure. In this section we derive the charged analogue of Whittaker's interior solution which has $(\rho + 3p) = G$ (constant) and reduces to the original neutral solution on the removal of charges.

The Whittaker's interior solution is expressed by the metric

$$ds^2 = -a(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + b(r)dt^2, \tag{1}$$

where

$$a(r) = \left[(1+B)(1-\alpha^2r^2) - B(\alpha r)^{-1}(1-\alpha^2r^2)^{3/2}\sin^{-1}\alpha r\right]^{-1} \tag{2}$$

and

$$b(r) = n\left[1 + B - B(\alpha r)^{-1}(1-\alpha^2r^2)^{1/2}\sin^{-1}\alpha r\right]. \tag{3}$$

Obviously,

$$ab = n(1-\alpha^2r^2)^{-1}. \tag{4}$$

The expressions for pressure and energy density are given by

$$8\pi p = \alpha^2 + B\alpha^2(\alpha r)^{-1}(1-\alpha^2r^2)^{1/2}\sin^{-1}\alpha r \tag{5}$$

and

$$8\pi\rho = 8\pi G - 3(8\pi p). \tag{6}$$

Positivity of p and ρ at the center demands

$$2\alpha^2 \le 8\pi G \le 6\alpha^2 \tag{7}$$

where B, α and n are constants.

It is easy to see that the gradient of pressure is negative which indicates that the pressure decreases when r increases. The pressure vanishes at $r = R$, where $y = \alpha R$ is the root of the equation

$$\alpha^2 y = 4\pi G(1-y^2)^{1/2}\sin^{-1} y. \tag{8}$$

It is clear that y is to lie between 0 and 1.

The continuity of the solution with the Schwarzschild's exterior solution at interface $r = R$ demands

$$n = 1 - \alpha^2R^2 \tag{9}$$

and

$$1 - \frac{2m}{R} = 8\pi G(1-\alpha^2R^2)\big/2\alpha^2. \tag{10}$$

3 Charged Analogue of the Whittaker's Solution

For the charged matter, the Einstein-Maxwell equations are given by

$$R^i_j - \frac{1}{2}R\delta^i_j = -8\pi T^i_j = -8\pi\left[M^i_j + E^i_j\right] \tag{11}$$

where M^i_j and E^i_j correspond to material and electromagnetic tensors respectively.

The non-vanishing components of the later can be given as

$$M^1_1 = M^2_2 = M^3_3 = -p \quad \text{and} \quad M^4_4 = \rho \tag{12}$$

as shown by Tolman [16],

$$E^1_1 = -E^2_2 = -E^3_3 = E^4_4 = -\frac{1}{2}g^{11}g^{44}F^2_{14} \tag{13}$$

as shown by Eddington [7] and also

$$J^4 = (-g)^{1/2}\frac{\partial}{\partial r}\left[-g^{11}g^{44}F_{14}\sqrt{-g}\right] \tag{14}$$

where p and ρ are pressure and energy density, while F_{14} and J^4 are non-vanishing components of electromagnetic field tensor and current vector respectively. J^4 as such, is accepted to be charge-density ($J^4 \cdot \sqrt{-g}$ being proper charge density).

The field equation (11) with reference to (1), (12) and (13) give

$$8\pi T^1_1 = -8\pi p + E = -\frac{b'}{abr} + \left(1 - \frac{1}{a}\right)\left(\frac{1}{r^2}\right), \tag{15}$$

$$8\pi T^2_2 = 8\pi T^3_3 = -8\pi p - E = \frac{1}{2ab}\left(\frac{ba'}{ar} - b'' + \frac{b'^2}{2b} + \frac{a'b'}{2a} - \frac{b'}{r}\right), \tag{16}$$

$$8\pi T^4_4 = 8\pi\rho + E = \frac{a'}{a^2 r} + \left(1 - \frac{1}{a}\right)\left(\frac{1}{r^2}\right), \tag{17}$$

where $E = -4\pi g'' g^{44} F^2_{14}$. Also

$$E = 4\pi(T^1_1 - T^2_2), \tag{18}$$

$$p = -\frac{(T^1_1 + T^2_2)}{2}, \tag{19}$$

and

$$\rho = T^4_4 - \frac{(T^1_1 - T^2_2)}{2}. \tag{20}$$

In order to derive the charged analogue of the Whittaker's interior solution, we shall solve the Einstein-Maxwell equations (15)–(17) for five unknowns, namely, $p, \rho, E, a(r)$, and $b(r)$. Therefore, we have freedom to assume two additional relations. Let us consider the expression for $b(r)$ to be the same as that of Whittaker's neutral solution and also the gravitational mass density $(\rho + 3p) = G$, a constant. So we have from (19), (20) and (15)–(17) and the relation assumed

$$T' - \left(\frac{2}{r} + \frac{3b'}{b}\right)T = \left(-8\pi Gr - \frac{2}{r}\right) \tag{21}$$

where $T = 1/a(r)$.

The above equation can always be satisfied by the metric potentials $a(r)$ and $b(r)$ of Whittaker's original solution. Therefore, the general solution of the above equation can be written as

$$T = \frac{1}{a(r)} + Ae^{\int(\frac{2}{r}+\frac{3b'}{b})dr}$$

or

$$T = \frac{1}{a(r)} + Ar^2b^3 \tag{22}$$

where $a(r)$ and $b(r)$ are given by (2) and (3). It is worth pointing out here that when $A = 0$, the $a(r)$ for the charged fluid will be identical to that of the perfect fluid case (2).

The expression for the energy momentum tensor for the charged fluid are furnished as below

$$8\pi T_1^1 = \left(\frac{1}{r^2}\right)\left(1 - \frac{b'r}{ab} - \frac{1}{a} - Ar^3b^2b' - Ar^2b^3\right), \tag{23}$$

$$8\pi T_2^2 = 8\pi T_3^3 = -\frac{1}{4b^2r}\left\{\left(\frac{1}{a} + Ar^2b^3\right)(2bb''r - b'^2r + 2bb') + b\left(\frac{a'}{a^2} + 2Arb^3 + 3Ar^2b^2b'\right)(2b + b'r)\right\}, \tag{24}$$

$$8\pi T_4^4 = \left(\frac{1}{r^2}\right)\left(1 - \frac{1}{a} + \frac{a'r}{a^2} - 3Ar^2b^3 - 3Ar^3b^2b'\right), \tag{25}$$

(18), (19) and (20) provide the expression for E, p, and ρ as below

$$E = \frac{Abr}{4}\left(rbb' + b^2\right)', \tag{26}$$

$$8\pi p = -\frac{1}{2r^2}\left(1 - \frac{1}{a}\right) + \frac{1}{4ar}\left(\frac{3b'}{b} - \frac{a'}{a}\right) + \frac{b''}{4ab} - \frac{b'}{8ab}\left(\frac{b'}{b} + \frac{a'}{a}\right) + A\left(\frac{7}{4}b^2b'r + b^3 + \frac{b^2b''r^2}{4} + \frac{bb'^2r^2}{4}\right), \tag{27}$$

$$8\pi\rho = \frac{1}{r^2}\left(1 - \frac{1}{a}\right) + \frac{a'}{a^2r} - \frac{Ab}{4}\left(12b^2 + 15bb'r + (bb')'r^2\right). \tag{28}$$

The expressions (26)–(28) on substituting for $a(r)$ and $b'(r)$ from (2)–(4) assume the following form

$$ER^2 = Q\left[N(1 + B\alpha^2R^2Y^2)^2 - b(1 - B\alpha^2R^2Y^2 + 2B\alpha^4R^4Y^4)\right], \tag{29}$$

$$8\pi pR^2 = B\alpha^2R^2 - \frac{b\alpha^2R^2}{N} + Q\left[N(1 + B\alpha^2R^2Y^2)^2 - b(1 - B\alpha^2R^2Y^2 + 2B\alpha^4R^4Y^4)\right] + \frac{AR^2b^2}{1 - \alpha^2R^2Y^2}\left[N(1 + B\alpha^2R^2Y^2) - b\alpha^2R^2Y^2\right], \tag{30}$$

$$8\pi\rho R^2 = -B\alpha^2 R^2 + \frac{3b\alpha^2 R^2}{N} - Q\big[N(1+B\alpha^2R^2Y^2)^2 - b(1 - B\alpha^2R^2Y^2 + 2B\alpha^4R^4Y^4)\big] - \frac{3AR^2b^2}{1-\alpha^2R^2Y^2}\big[N(1+B\alpha^2R^2Y^2) - b\alpha^2R^2Y^2\big], \tag{31}$$

where $Y = r/R$, $Q = \frac{AR^2Nb}{4(1-\alpha^2R^2Y^2)^2}$.

The solution so obtained is joined smoothly to the Nordstörm metric,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{4\pi\varepsilon^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + \left(1 - \frac{2M}{r} + \frac{4\pi\varepsilon^2}{r^2}\right) dt^2 \tag{32}$$

at pressure free interface $r = R$ and we get the following conditions to be imposed upon the constants

$$F_{14}^2 = \frac{\varepsilon^2}{R^4} = \frac{E}{4\pi}, \tag{33}$$

and $M = 4\pi\int_0^R (\rho + \frac{E}{8\pi})r^2dr$,

$$\frac{1}{a_R} = b_R = \left(1 - \frac{2M}{R} + \frac{4\pi\varepsilon^2}{R^2}\right), \tag{34}$$

$$p_R = 0. \tag{35}$$

Now solving the relations (33)–(35) using (29), (30), (2) and (3) we get the following values of the constants A, B, N and α:

$$AR^2 = \left(\frac{1}{b_R^2}\right)\left[\frac{\{(2-X)\alpha^2R^2 - 1\}b_R + (1-\alpha^2R^2)(X\alpha^2R^2 + 1 - \frac{2M}{R})}{\{(2-X)\alpha^2R^2 - 1\}b_R + (1-\alpha^2R^2)(2 - \frac{2M}{R})}\right], \tag{36}$$

$$B = \left[\frac{X(1-\alpha^2R^2)(2b_R - 2 - \frac{2M}{R})}{\{(2-X)\alpha^2R^2 - 1\}b_R + (1-\alpha^2R^2)(2 - \frac{2M}{R})}\right], \tag{37}$$

$$N = \left[\frac{\{(2-X)\alpha^2R^2 - 1\}b_R + (1-\alpha^2R^2)(2 - \frac{2M}{R})}{(1 - X\alpha^2R^2)}\right], \tag{38}$$

where $X = [1 - (1-\alpha^2R^2)^{1/2}\frac{\sin^{-1}\alpha R}{\alpha R}]^{-1}$ and α is given by the equation (35). Also the expression for J^4 can be obtained from (14) and assumes the form as below

$$J^4R^2 = \left(Y^2\sqrt{4\pi a^* b}\right)^{-1}\left[2Y\sqrt{ER^2} + Y^2\left(2\sqrt{ER^2}\right)^{-1}\frac{\partial(ER^2)}{\partial Y}\right],$$

where $\frac{1}{a^*} = \frac{b}{N}[(1-\alpha^2R^2Y^2) + (AR^2)Y^2b^3]$.

The above solution has been analyzed numerically and the limitations on the various physical quantities are discussed.

The positivity of p, ρ and the reality condition $\rho \geq p$ at the origin restrict the constant as

$$1 < \frac{\alpha^2R^2B}{\alpha^2R^2 - AR^2N^3} < 2.$$

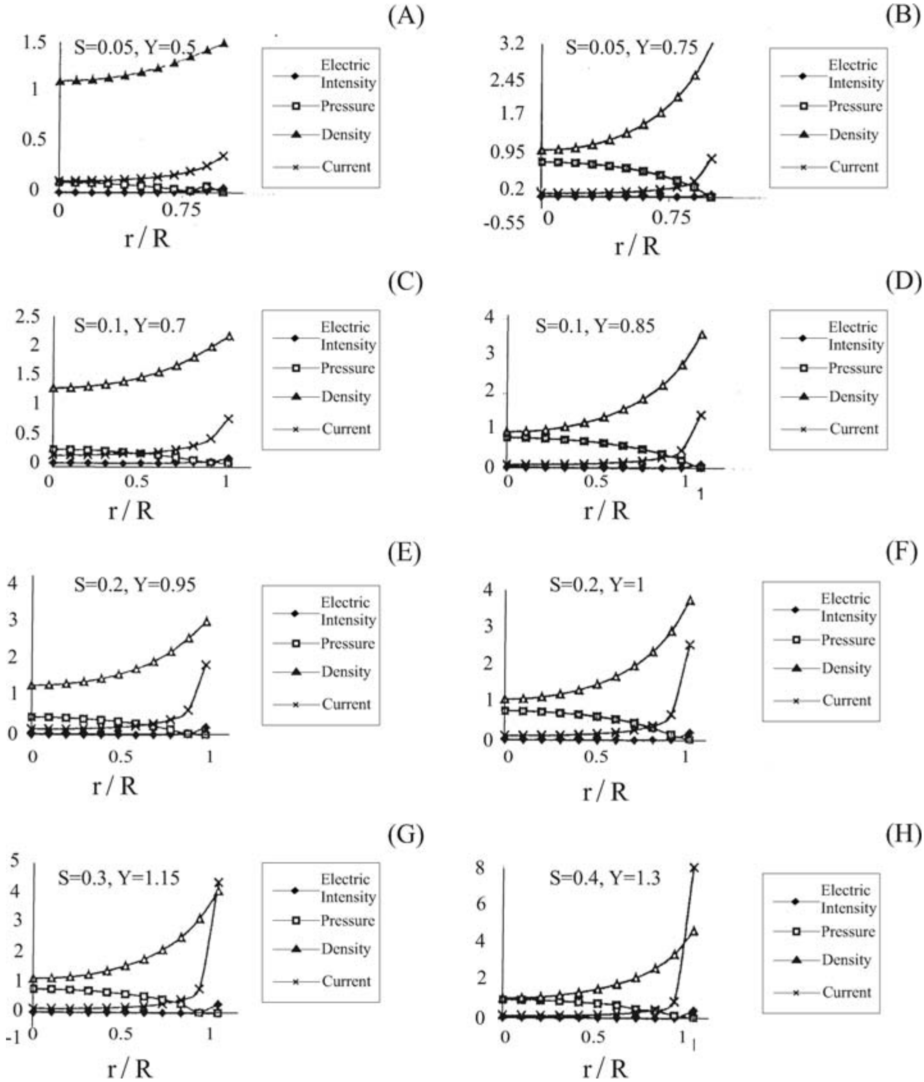

Fig. 1 The behaviour of various physical quantities with r/R

4 Numerical Results for the Charged Solution

Numerical values of various physical quantities like energy density, pressure, electric intensity and J^4 are obtained numerically by considering the various sets of $S = \frac{4\pi\varepsilon^2}{R^2}$ and $Y = \frac{2M}{R}$ and plotted in Figure 1. The procedure involves the solution of the equation $p_R = 0$ for a given set of S and Y. In case, root (value of αr) exists we calculate the values of the said physical quantities. Roots of the equation $p_R = 0$ are found to exist for the following possible combinations of S and Y (Table 1). The range of Y is also displayed for which the density, pressure, electric intensity and J^4 are positive and satisfying the reality condition such as $\rho > p$. It is observed that the physical quantities satisfying the reality conditions are not possible for $S > 0.4$ at any value of Y subject to the division $S = 0$ (0.5) 0.8 and

Table 1 Range of S and Y for which roots of $p_R = 0$ exist

S	Y (for physically valid quantities)
0.00	0.05–0.65
0.05	0.50–0.85
0.10	0.70–0.85
0.15	0.85–0.90
0.20	0.95–1.00
0.25	1.05
0.30	1.15
0.35	1.205
0.40	1.30

$Y = 0.05\ (0.5)\ 1.75$. In the following table, the range of Y and S for which the roots of (35) exist, are displayed. Further the second column contain the interval of those roots for which all the physical quantities like energy density, pressure, electric intensity and J^4 are physically valid. It is worth mentioning here that by roots, we mean the radius of isolated charged fluid sphere for which the pressure is vanishing at the boundary. Besides the above observations, a comparative behavior of various physical quantities can easily be seen on the graphs of pressure p, energy density ρ, electrostatic energy density E and charge density J^4 (Fig. 1).

5 Concluding Remarks

We derive first ever charged analogue of Whittaker's interior solution joining smoothly to Reissner-Nordstörm solution. Moreover, the comparative behavior of various physical quantities is also analysed and displayed on graphs.

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